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The Membrane Picture of Hawking Radiation

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The effect of a black hole on the classical physics of exterior electromagnetic fields can be expressed by replacing the black hole by a conducting membrane. We show that when we introduce quantum mechanics the currents in this membrane must also satisfy a quantum Langevin equation and that this, together with the nonzero transmission coefficient for the potential barrier around the hole in the membrane picture, gives rise to the Hawking radiation.

KEY WORDS: Black holes; Hawking radiation; stretched horizon.

1. INTRODUCTION

The treatment of a black hole as a one-way membrane has been shown to be of considerable use in providing a physical picture of the properties of black holes. This is achieved, at least in part, by the 3 + 1 splitting of space-time in the membrane picture which makes possible the use of Newtonian concepts. This is analogous to the usual way we use the splitting of the electromagnetic field in Maxwell's equations to provide a Newtonian framework in flat spacetime. So far, for black holes, although it has provided some insight into quantum phenomena, this splitting has proved most useful in classical physics. This is somewhat paradoxical, since the 3 + 1 approach fits naturally into a Hamiltonian context.

The physical picture we expect to emerge from the membrane treatment of black hole evaporation is of radiation from a surface at the Hawking temperature according to the normal laws of thermal physics. Such a picture might be expected to provide a consistent explanation of the following: (i) The *emission* of radiation from the hole: the often quoted

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heuristic model of the separation of virtual particle–antiparticle pairs by curvature forces, for example, is not viable.[4] (ii) The absence of radiation from a uniformly accelerated conductor in the Minkowski vacuum. We shall see below that the membrane which replaces the black hole in the $3+1$ picture is endowed with the properties of an electrical conductor. By the equivalence principle such a conductor is locally equivalent to one in uniform acceleration in the absence of gravity, yet the former must emit Hawking radiation. (iii) Why the horizon is at a nonzero temperature. This is probability the hardest requirement to satisfy. Of course, one can appeal to quantum field theory to provide this information, but it would be more satisfying to translate the result into the membrane physics.

In addition, one might expect to show that the membrane behaves consistently in response to departures from thermal equilibrium (or, equivalently, that the picture is consistent with fluctuations at equilibrium.) In fact, the main point of this paper will be to set out the physical model and to demonstrate this last point. We shall also suggest answers to the some of the other problems. We shall restrict discussion to external electromagnetic fields so the membrane geometry is fixed.

2. THE MEMBRANE AS A DISSIPATIVE SYSTEM

The stretched horizon of a black hole is defined by mapping each point of the event horizon back along a past-directed radial null geodesic to a sphere the area of which is a prescribed amount ε greater than that of the horizon itself. The quantity ε is assumed small but is otherwise undefined; Thorne *et al.*[8] assume ε is proportional to the horizon area at the base point of the null ray, whereas Susskind *et al.*[7] take ε constant and equal to a Planck length. The latter approach has the advantage that the stretched horizon is at a fixed value of the 'tortoise' radial coordinate $r_* = r_0$, say; this is the version we shall adopt, although we shall not need to specify a value for r_0. [We shall use the radial coordinate r_* rather than the Schwarzschild coordinate r; the two are related by $r_* = r + (2GM/c^2)\ln(c^2r/2GM - 1)$ for a hole of mass M.]

This stretched horizon is then endowed with properties of a membrane that represents the effect of the hole as far as external fields are concerned. The hole is therefore replaced by a physical timelike surface. For example, for the electromagnetic field the stretched horizon must act as a terminator of normal components of electric and magnetic fields. It must therefore be capable of acquiring charge and current distributions; i.e., it must act as a conductor. The physical requirement that freely falling observers see finite fields at the horizon translates to the requirement that the membrane fields (i.e., the fields $\mathbf{E}_H$, $\mathbf{B}_H$ as measured by an observer in the stretched horizon) must appear as ingoing waves. Hence $\mathbf{E}_H = \mathbf{n} \wedge \mathbf{B}_H$, where $\mathbf{n}$ is a unit (3-vector) normal to the membrane.

From Ampere's law,

$$\int \mathbf{B}_H \cdot \mathbf{dl} = 4\pi \int \mathbf{j}_H \cdot \mathbf{dA} \tag{1}$$

we define a surface current density (per unit length) $\mathbf{J}_H$ flowing in the membrane by taking the membrane to have a thickness δ, so $\mathbf{dA} = \delta \mathbf{n} \wedge \mathbf{dl}$. Then

$$\mathbf{B}_H = 4\pi \mathbf{J}_H \wedge \mathbf{n} \tag{2}$$

defines the surface current density $\mathbf{J}_H$. Expressing the boundary condition in terms of $\mathbf{J}_H$ gives

$$\mathbf{J}_H = (1/4\pi)\, \mathbf{E}_H \tag{3}$$

Thus the stretched horizon acts as a membrane with resistivity $1/4\pi$ in units in which $c = 1$.[9, 2]

If we consider time-dependent fields, we can find also the self-inductance of the hole. We have from Ampere's law

$$\dot{\mathbf{B}}_H = 4\pi \dot{\mathbf{J}}_H \wedge \mathbf{n} \tag{4}$$

We now apply Faraday's law to a closed circuit in the $\phi = \text{const}$ plane made up of an arc in the membrane, two radial spokes, and an arc at $r = \text{const}$ at a large radius. The emf in this circuit is

$$\mathscr{E} = -\int \mathbf{E}_H \cdot \mathbf{dl} = \frac{1}{\alpha} \frac{d}{dt} \int \mathbf{B} \cdot \mathbf{dA} \tag{5}$$

where $\alpha = (1 - 2M/r)^{1/2}$ for a hole of mass M in geometrical units ($G = 1$, $c = 1$). To relate $\dot{\mathbf{B}}$ to the current flow we imagine conical probes to carry a total current I to and from the poles of the stretched horizon with a uniform current density in the horizon. The solution for the fields in this case is known.[2] We have

$$B_\phi = \frac{-2I}{\alpha r \sin\theta} \tag{6}$$

Thus $\mathscr{E} \propto \dot{B}_\phi \propto \dot{I}$, the final coefficient of proportionality being the self-inductance L. (If the current changes on a time scale Δt, then the outer arc of the circuit must be at a distance less than $c\,\Delta t$ in order that the situation be steady. This means that what we are really calculating is the inductance of the black hole in its environment; the point is not the value of L, but the fact that there is an induction effect associated with a changing surface current.)

The surface current density $\mathbf{J}_H$ represents the internal degrees of freedom of the black hole in its interaction with the external electromagnetic field. We can think of the total current I as a collective degree of freedom. In the presence of an external potential, I satisfies a Langevin equation:

$$L\dot{I} + 4\pi I = V \tag{7}$$

In classical physics the ground state of the system is $V=0$, $I=0$. In the quantum ground state I must contain vacuum fluctuations, so $\langle I \rangle = 0$ but $\langle I^2 \rangle \neq 0$. But since the system is dissipative, such fluctuations would decay unless maintained by a fluctuating force, here the potential difference V. It is the Joule heating driven by V that is the Hawking radiation (for the electromagnetic field) (compare Susskind *et al.*[1]).

The Langevin equation (7) is in fact a remarkable result. In a microscopic theory the coupling of a (finite) quantum system to a field leads to a Langevin equation in which the dissipation rate coefficient is related to the coupling: for example, for a harmonic oscillator of unit mass with coordinate q in scalar electrodynamics we have (e.g., Raine *et al.*[6])

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = -e\dot{\phi} \tag{8}$$

where $\gamma = e^2/2$. This then leads to a fluctuation-dissipation theorem: in terms of the susceptbility $\chi(\omega) = 1/(-\omega^2 + \omega_0^2 - i\omega\gamma)$ this can be expressed as $\operatorname{Im}\chi = \frac{1}{2}e^2\omega\,|\chi|^2$ (e.g., Massar *et al.*[5]). For a phenomenological Langevin equation there is no reason why γ should be such that this relation will hold. Yet, for the Langeving equation (7) the resistivity of the hole has turned out to be precisely what is required to fulfill a relation of this type. (The extra factor of 4π comes—correctly—from the three-dimensionality.) This suggests that (7) is not merely a phenomenological relation, but should be derivable from a microscopic theory, i.e., from a quantum theory of gravity.

3. THE THERMAL MEMBRANE

If $\langle V^2 \rangle$ is thermal at the Hawking temperature T_H, then from (7) the membrane will satisfy the Nyquist relation, i.e., will behave as a resistor at T_H and radiate thermally into its surroundings. The most difficult part of the problem to understand without recourse to quantum field theory is why the field at the horizon should be thermal. We can only supply a heuristic argument, but one that will raise an important problem. To obtain a more rigorous derivation, we should look at the dynamical evolution of the horizon, a matter to which we hope to return elsewhere.

At late times a black hole formed from collapse is indistinguishable from an eternal hole. In the latter case an observer at rest on the stretched horizon at r_0 is locally equivalent to a constantly accelerated observer in Minkowski spacetime with acceleration equal to the surface gravity. Both observers should therefore see a thermal bath at the Hawking temperature.

If this equivalence is valid, one can ask the question: why does the stretched horizon radiate, whereas a constantly accelerated conductor in Minkowski spacetime does not? The reason is related to the different potential barriers for the fields between the conductors and spatial infinity in the two cases.

The external field can be shown to be derivable from an equation of the form

$$\frac{d^2\phi}{dr_*^2} + V(r_*)\phi = 0 \tag{9}$$

(e.g., Chandrasekhar[1] and Futterman *et al.*[3]). The asymptotic forms at r_0 near the hole ($r_* \to -\infty$) and at spatial infinity are given by

$$\phi = e^{-ikr_*} + Ae^{ikr_*} \tag{10}$$

$$= Be^{ikr_*} \tag{11}$$

Next we need a simple result for solutions of equations of the form (9). Write the equation as $(L+V)\phi = 0$. Then the scattered field asymptotically, $\psi = \phi + L^{-1}V\phi$, is propagated by the free field propagator, since $L\psi = 0$. In particular, since $V(r_*) \to 0$ as $r_* \to \pm\infty$, the asymptotic fields (11) are propagated by the free Green function, which, near r_0, is the flat-space one (in terms of r_*).

Now, if G is a Green function for the external field equation (9), the field in the presence of the membrane with current density J_H is of the symbolic form

$$\phi = \phi_0 + \int GJ_H\, dA \tag{12}$$

The noise power as a result of the membrane will change by (compare Raine *et al.*[6])

$$\langle \phi(x)\,\phi(x')\rangle - \langle \phi_0(x)\,\phi_0(x')\rangle = \left\langle \phi_0 \int GJ \right\rangle + \left\langle \int GJ\phi_0 \right\rangle + \left\langle \int GJ \int GJ \right\rangle \tag{13}$$

where the Fourier transforms of the currents are given by

$$J_\omega = \chi_\omega \phi_\omega \tag{14}$$

To obtain the change in noise power we use the result of Masser *et al.*[5]: omitting the Casimir terms and using ϕ_L and ϕ_R for the left- and right-moving components, respectively their Eq. (27) can be written as the Fourier transform of

$$|\chi_\omega| \left[\langle \phi_R(x)\, \phi_R(x')\rangle + \langle \phi_L(x)\, \phi_L(x')\rangle\right] + (i\chi_\omega - i\chi_\omega^*)\langle \phi_L(x)\, \phi_L(x')\rangle \tag{15}$$

Substituting for ϕ_L and ϕ_R from (11) near r_0 gives a thermal factor times

$$|\chi|^2 (1 + |A|^2) + i(\chi - \chi^*)\, |A|^2 = |A|^2 \left[2\, |\chi|^2 + i(\chi - \chi^*)\right] + (1 - |A|^2)\, |\chi|^2 \tag{16}$$

The first term in square brackets on the right vanishes by the fluctuation dissipation theorem for the membrane; the final term represents the emission of energy by the membrane. In fact since $1 - |A|^2 = |B|^2$, this is just a thermal spectrum modified by a transmission factor, as expected.

For an accelerated perfect conductor in flat spacetime we have $A = 1$ and no emission (compare Raine *et al.*[6] and Massar *et al.* [5]).

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